

# Monodromy representations of the braid group <sup>1</sup>

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## Abstract

Chiral conformal blocks in a rational conformal field theory are a far going extension of Gauss hypergeometric functions. The associated monodromy representations of Artin's braid group  $\mathcal{B}_n$  capture the essence of the modern view on the subject, which originates in ideas of Riemann and Schwarz.

Physically, such monodromy representations correspond to a new type of braid group statistics, which may manifest itself in two-dimensional critical phenomena, e.g. in some exotic quantum Hall states. The associated primary fields satisfy  $R$ -matrix exchange relations. The description of the internal symmetry of such fields requires an extension of the concept of a group, thus giving room to quantum groups and their generalizations.

We review the appearance of braid group representations in the space of solutions of the Knizhnik–Zamolodchikov equation, with an emphasis on the role of a regular basis of solutions which allows us to treat the case of indecomposable representations of  $\mathcal{B}_n$  as well.

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<sup>1</sup> Plenary talk, presented by I.T. Todorov at the XXIII International Colloquium on Group Theoretical Methods in Physics, Dubna, July 31 – August 5, 2000

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# Introduction

Artin's braid group  $\mathcal{B}_n$  – with its monodromy representations – is a good example of a focal point for important developments in both mathematics and physics.

In *mathematics* it appears in the description of topological invariants of algebraic functions [1] and the related study of multiparametric integrals and (generalized) hypergeometric functions [2 – 4] as well as in the theory of knot invariants and invariants of three-dimensional manifolds [5 – 9]. The main *physical applications* go under the heading of *generalized statistics* (anticipated by Arnold in [1] – see Section 1). The *Knizhnik–Zamolodchikov* (KZ) *equation* (Section 2) is a common playground for physicists and mathematicians.

We illustrate highbrow mathematical results of [10] and [11] on the relation between the monodromy representations of  $\mathcal{B}_n$  in the space of solutions of the KZ equation, for a semisimple Lie algebra  $\mathcal{G}$ , and the universal  $R$ -matrix for  $U_q(\mathcal{G})$  by simple computations for the special case of  $\mathcal{G} = su(N)$  step operators and  $\mathcal{B}_3$  (Section 3). In fact, we go in our explicit construction beyond these general results by also treating on an equal footing the indecomposable representations of  $\mathcal{B}_3$  for  $q$  an even root of unity ( $q^h = -1$ ).

## 1 Permutation and braid group statistics

The symmetry of a one-component wave function  $\Psi(x_1, \dots, x_n)$  is described by either of the one-dimensional representations of the group  $\mathcal{S}_n$  of permutations giving rise to Bose and/or Fermi statistics. Multicomponent wave functions corresponding to particles with internal quantum numbers, may transform under higher dimensional representations of  $\mathcal{S}_n$  corresponding to parastatistics. If one allows for multivalued wave functions, then the exchange of two arguments  $x$  and  $y$  may depend on the (homotopy type of the) path along which  $x$  and  $y$  are exchanged, giving rise to a representation of the *braid group*  $\mathcal{B}_n$  of  $n$  *strands*.

$\mathcal{B}_n$  was defined by E. Artin in 1925 as a group with  $n - 1$  generators,  $b_1, \dots, b_{n-1}$  ( $b_i$  “braiding” the strands  $i$  and  $i + 1$ ), satisfying the following two relations:

$$b_i b_j = b_j b_i, \quad |i - j| \geq 2, \quad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad i = 1, \dots, n - 2. \quad (1.1)$$

Let  $\sigma : \mathcal{B}_n \rightarrow \mathcal{S}_n$  be the group homomorphism defined by

$$\sigma(b_i) = \sigma_i, \quad \sigma_i^2 = 1 (\in \mathcal{S}_n), \quad i = 1, \dots, n - 1, \quad (1.2)$$

where  $\sigma_i$  are the basic transpositions exchanging  $i$  and  $i + 1$  that generate  $\mathcal{S}_n$ . The kernel  $\mathcal{P}_n$  of this homomorphism is called the *monodromy* (or pure braid) *group*. Note that the element

$$c^n := (b_1 b_2 \dots b_{n-1})^n \quad (1.3)$$

generates the centre of  $\mathcal{B}_n$ .

The braid group  $\mathcal{B}_n$  and its invariant subgroup  $\mathcal{P}_n$  have a topological interpretation. Consider the  $n$ -dimensional manifold

$$Y_n = \mathbb{C}^n \setminus \text{Diag} \equiv \{ \vec{z} = (z_1, \dots, z_n) \in \mathbb{C}^n; \quad i \neq j \Rightarrow z_i \neq z_j \} \quad (1.4)$$

( $Y_n$  is the analyticity domain of  $n$ -point conformal blocks in chiral conformal field theory). The symmetric group  $\mathcal{S}_n$  acts on  $Y_n$  by permutations of coordinates. The factor space  $X_n = Y_n/\mathcal{S}_n$  is the *configuration space of  $n$  points* (“identical particles”) in  $\mathbb{C}^n$ .

**Proposition 1.1** ([1]) *The braid group  $\mathcal{B}_n$  is isomorphic to the fundamental group  $\pi_1(X_n, \vec{z}_0)$  of the configuration space (for, say,  $\vec{z}_0 = (n, \dots, 1)$ ); similarly,  $\mathcal{P}_n \simeq \pi_1(Y_n, \vec{z}_0)$ .*

Clearly, had we substituted the complex plane,  $\mathbb{C} \simeq \mathbb{R}^2$ , by an  $s$ -dimensional space  $\hat{R}^s$  for  $s \geq 3$ , the fundamental group  $\pi_1((\hat{R}^s)^{\otimes n} \setminus \text{Diag}, \vec{z}_0)$  would have been trivial, and no interesting relation with the braid group could be expected. This simple topological fact explains why the possibilities for generalized statistics are richer in low (one- and two-) dimensional physics. One may wonder why it took more than half a century after the appearance of Bose and Fermi statistics in quantum mechanics for such a basic observation to gradually find its way into the physics literature. The notion of a braid group seems to have first appeared in physics in connection with the Dirac string [12]; the peculiarity of low dimensional statistics was noted in [13]; the problem was then treated more systematically in the framework of quantum mechanics [14] and in the context of local current algebras [15]. Moreover, this pioneering work did not attract much attention before it was repeated by others (starting with [16]) when catch-words such as “anyons” were coined. The story of the ancestry of the “anyon” has once been told with authority [17], but it continues to be ignored in “mainstream publications”.

A deeper understanding of particle statistics came from the “algebraic” study of superselection sectors in local quantum field theory (see [18] and [19] where earlier work of Doplicher, Haag and Roberts is also cited). We offer here an informal (physicist’s oriented) formulation of the main result of this work.

The starting point of the *algebraic* (Haag–Kastler) *approach* is the concept of an *algebra  $\mathcal{A}$  of local observables*. As lucidly explained in [20], it provides an intrinsic, coordinate-free description of the algebra generated by local gauge invariant (Wightman) Bose fields – such as the stress-energy tensor and conserved  $U(1)$  currents. Departing slightly from the purist’s algebraic view, we shall identify from the outset  $\mathcal{A}$  with its *vacuum representation* in a Hilbert space  $\mathcal{H}$  that carries a unitary positive energy representation of the Poincaré group with a *unique* translation invariant *vacuum state*. It is important that gauge-dependent charge carrying (and/or multivalued) fields are excluded from  $\mathcal{A}$ . They reappear – as derived objects – in the role of intertwiners among inequivalent representations of  $\mathcal{A}$ .

*Superselection sectors* are defined by *irreducible positive energy representations* of  $\mathcal{A}$  that can be obtained from the *vacuum sector* by the action of *localizable “charged fields”* – i.e. of fields that commute at space-like separations with the observables (but need not be local among themselves). Products of charged fields acting on the vacuum give rise, typically, to a finite sum of superselection sectors defining the *fusion rules* of the theory. (To make this precise one needs, in fact, more elaborate tools – such as \*-endomorphisms of a completion of  $\mathcal{A}$  that are localizable in space-like cones; the shortcoming of a simple-minded use of “charged fields” is the non-uniqueness of their choice and hence, of the multiplicities entering the above naive definition of fusion rules.) A fancy way to express

the fact that there is a well defined composition law for representations of  $\mathcal{A}$  (analogous to tensor product of group representations) is to say that *superselection sectors give rise to a tensor category*. A memorable result of Doplicher and Roberts [19] crowning two decades of imaginative work of Haag’s school says that, for a local quantum theory with no massless excitations in space-time dimensions  $D \geq 4$ , this category is equivalent to the category of irreducible representations (IR) of a compact group  $G$ . In more down-to-earth terms it means that  $G$  acts – by automorphisms – on charged fields as a *gauge group of the first kind* (recall that a gauge group leaves all observables invariant, not only the Hamiltonian). Superselection sectors are labelled by (equivalence classes of) IR  $p \in \hat{G}$  (borrowing the terminology of representation theory of semisimple compact Lie groups, we shall call the labels  $p$  *weights*). The state space of the theory can thus be viewed as a direct sum of tensor product spaces:

$$\mathcal{H} = \bigoplus_{p \in \hat{G}} \mathcal{H}_p \otimes \mathcal{F}_p, \quad d(p) := \dim \mathcal{F}_p < \infty, \quad (1.5)$$

where  $\mathcal{F}_p$  are irreducible  $G$ -modules. The statistics of a sector  $p$  is characterized by a *statistics parameter*  $\lambda_p = \pm \frac{1}{d(p)}$ . If  $G$  is abelian (the common case of commuting superselection sectors labelled by the “spin parity”  $e^{2\pi i s_p}$ , where  $s_p$  is the spin, and by the values of the electric, baryonic and leptonic charge), then  $d(p) = 1$  for all sectors and we are faced with the familiar *Bose–Fermi alternative*. If  $G$  is non-abelian and  $d(p) = 2, 3, \dots$ , then the sector  $p$  and its conjugate  $\bar{p}$  (or, in physical language, the particles of type  $p$  and their antiparticles  $\bar{p}$ ) obey parastatistics. (Unfortunately, one has no such result for quantum electrodynamics. It is, in fact, known that the electric charge cannot be localized in a space-like cone. Although there is no indication that, say, electrons may obey braid group statistics, we are unable to rule it out at present.)

These results also extend to space-time dimension  $D = 3$ , provided the superselection charges can be localized in finite regions. In more realistic  $(2 + 1)$ -dimensional systems (like a “quantum Hall fluid” in a strong magnetic field perpendicular to the plane of the layer), charges can be localized only in infinite space-like cones and there is room for braid group statistics. For  $D = 2$  braid group statistics may appear even for (superselection) charges localized in finite domains (see [21 – 23] and references to earlier work of these authors cited there). The notion of a statistics parameter extends to this case, too, and is related to Jones index of inclusion of associated factors of operator algebras [22]. It can be written as (see [23], Definition 6.2)

$$\lambda_p = \frac{1}{d(p)} e^{-2\pi i \theta_{p\bar{p}}}, \quad d(p) (= |\lambda_p|^{-1}) > 0, \quad e^{-2\pi i \theta_{p\bar{p}}} = e^{-i\pi(s_p + s_{\bar{p}})}, \quad (1.6)$$

where  $s_p$  and  $s_{\bar{p}}$  are the (fractional) spins of the conjugate sectors  $p$  and  $\bar{p}$ . For  $d(p) = 1$ ,  $\lambda_p \neq \pm 1$ , we are dealing with a one-dimensional representation of the braid group, corresponding to *anyonic statistics*. For non-integer  $d(p)$  the “gauge symmetry” of superselection sectors cannot be described by a group. In fact, soon after quantum groups were introduced [24] the attractive possibility of applying them to describing the symmetry of two-dimensional (conformal) models was considered by several authors [25 – 32]

(refs. [26, 27] appearing as predecessors of both [21 – 23] and [19]). These first attempts did not face the problem of incompatibility between Wightman (Hilbert space) positivity and manifest quantum group invariance. Three different approaches have been developed to deal with this problem in terms of: (i) weak quasiHopf algebras [33]; (ii) weak co-associative star Hopf algebras [34] or quantum groupoids [35] related recently to the Ocneanu “double triangle algebra” and to boundary conformal field theory [36]; (iii) a BRS approach with quantum group symmetry in an extended state space [37, 38]. None of these developments has been fully conclusive.

To cite [23], “braid statistics in two-dimensional systems is more than a theoretical curiosity”. Indeed, anyons have made their way in the standard interpretation of the fractional Hall effect. Non-abelian braid group statistics appears to be strongly indicated for Hall plateaux at the second Landau level with filling fractions  $\nu = 2 + \frac{m}{m+2}$ ,  $m = 2, 3, \dots$  (cf. [39, 40]).

## 2 The KZ equation

Let  $\mathcal{G}$  be a compact Lie algebra,  $V$  a finite-dimensional  $\mathcal{G}$  module, and  $C_{ab}$  the (polarized) Casimir invariant acting nontrivially on the factors  $a$  and  $b$  of the  $n$ -fold tensor product  $V^{\otimes n}$ . For  $\mathcal{G} = su(N)$  and  $n = 3$  we have

$$C_{12} (= C_{21}) = \left( \sum_{i,j=1}^N e_{ij} \otimes e_{ji} - \frac{1}{N} \sum_{i=1}^N e_{ii} \otimes \sum_{j=1}^N e_{jj} \right) \otimes \mathbb{1}, \quad (2.1)$$

where  $e_{ij}$  represent the Weyl generators of  $U(N)$  in  $V$ .

The KZ equation is a system of partial differential equations, which can be written compactly as

$$h d\Psi = \sum_{1 \leq a < b \leq n} C_{ab} \frac{dz_{ab}}{z_{ab}} \Psi, \quad z_{ab} = z_a - z_b, \quad dz_{ab} = dz_a - dz_b, \quad C_{ab} = C_{ba}; \quad (2.2)$$

here  $h$  is a (say, real) parameter,  $\Psi = \Psi(z_1, \dots, z_n)$  is a (regular) map,  $\Psi : Y_n \rightarrow V^{\otimes n}$ , where  $Y_n$  is  $\mathbb{C}^n$  minus the diagonal (see (1.4)). The system (2.2) has a nice geometric interpretation: it defines a *connection*  $\nabla = d - \Gamma$  on the trivial bundle  $Y_n \times V^{\otimes n}$ , where  $\Gamma$  is the connection 1-form

$$\Gamma = \frac{1}{h} \sum_{a < b} C_{ab} \frac{dz_{ab}}{z_{ab}}. \quad (2.3)$$

Introducing the corresponding *covariant derivatives*

$$\nabla_a = \frac{\partial}{\partial z_a} - \frac{1}{h} \sum_{b \neq a} \frac{C_{ab}}{z_{ab}}, \quad a = 1, \dots, n, \quad (2.4)$$

we can interpret (2.2) by saying that  $\Psi$  is covariantly constant. This requires as a compatibility condition the flatness of the KZ connection.

**Proposition 2.1** *The KZ connection  $\nabla = d - \Gamma$  has zero curvature:*

$$\nabla \circ \nabla = \Gamma \wedge \Gamma - d\Gamma = 0 \quad \Leftrightarrow \quad [\nabla_a, \nabla_b] = 0. \quad (2.5)$$

The *proof* (see, for example, [41]) uses

$$[C_{ab}, C_{cd}] = 0 \quad \text{for different } a, b, c, d, \quad (2.6)$$

$$[C_{ab}, C_{ac} + C_{bc}] = 0 = [C_{ab} + C_{ac}, C_{bc}] \quad \text{for different } a, b, c, \quad (2.7)$$

as well as the following *Arnold's lemma*: let

$$u_{ab} = d(\ln z_{ab}) = \frac{dz_{ab}}{z_{ab}} \left( \equiv \frac{dz_a - dz_b}{z_a - z_b} \right); \quad (2.8)$$

then

$$u_{ab} \wedge u_{bc} + u_{bc} \wedge u_{ca} + u_{ca} \wedge u_{ab} = 0 \quad \text{for } a \neq b \neq c \neq a. \quad (2.9)$$

The flatness of the connection  $\nabla$  is a necessary and sufficient condition for the holonomy group  $\mathcal{P}_n$  at a point  $p \in Y_n$  (the transformation group in  $V^{\otimes n}$  obtained by parallel transport of vectors along closed paths with beginning and end in  $p$ ) to give rise to a (*monodromy*) representation of the fundamental group  $\pi_1(Y_n, p)$ .

The KZ equation appears in 2D CFT in the context of the Wess–Zumino–Novikov–Witten (WZNW) [42] model [43] and in a related study of chiral current algebras [44]. The idea of the latter approach is simple to summarize. A *primary* field  $\varphi$  of a conformal current algebra is *covariant* under two (infinite-parameter) infinitesimal transformations: under local *gauge transformations* generated by the currents  $J$  and under *reparametrization* generated by the stress-energy tensor  $T$ . On the other hand,  $T$  is expressed quadratically in terms of  $J$  (by the so-called Sugawara formula). The consistency between the two covariances and this quadratic relation yields the *operator KZ equation*:

$$h \frac{d\varphi}{dz} = : \varphi(z) \vec{t} J(z) :. \quad (2.10)$$

Here  $C_{ab} = \vec{t}_a \otimes \vec{t}_b$ , the vector  $\vec{t}$  spanning a basis of the finite-dimensional representation of  $\mathcal{G}$  such that  $[\vec{J}_0, \varphi(z)] = \varphi(z) \vec{t}$  for  $\vec{J}_0 = \oint \vec{J}(z) \frac{dz}{2\pi i}$ , and the “height”  $h$  is an integer ( $h \geq N$  for  $\mathcal{G} = su(N)$ ). Using also the current-field Ward identity, we end up with Eq. (2.2) for the “wave function”

$$\Psi(p; z_1, \dots, z_n) = \langle p | \varphi(z_1) \otimes \dots \otimes \varphi(z_n) | 0 \rangle, \quad (2.11)$$

where  $p$  stands for the weight of the  $\mathcal{G}$ -module that contains the bra  $\langle p |$  (see Appendix B).

The notation in Eq. (2.11) is, in fact, ambiguous. There are, for fixed  $n$  and  $p$ , several linearly independent solutions (called *conformal blocks*) of the KZ equation. To distinguish between them one introduces the concept of a *chiral vertex operator* (CVO)

[25] (the counterpart of an intertwiner between different superselection sectors in the algebraic approach to local quantum field theory [19, 21 – 23]). We shall use instead a field  $\varphi$  belonging to the tensor product  $V \otimes \mathcal{V}$  of a  $\mathcal{G}$  and a  $U_q(\mathcal{G})$  module,  $\varphi = (\varphi_\alpha^A)$ ; it arises naturally in splitting the group valued field  $g$  in the WZNW model into left and right movers,  $g_B^A(z, \bar{z}) = \varphi_\alpha^A(z)(\bar{\varphi}^{-1})_B^\alpha(\bar{z})$  (see [45, 46] and [38] for two early and a recent paper, the latter containing some 50 more references on the subject). Take as an example  $\mathcal{G} = su(N)$ ,  $n = 3$  and  $\varphi$  an  $SU(N)$  *step operator* (i.e.  $V = \mathbb{C}^N$  carrying the defining representation of  $su(N)$ ). Then, if we take  $p$  to be the highest weight of the IR associated with the Young tableau  $\boxplus$  with respect to both  $su(N)$  and  $U_q(sl_N)$ , we can reduce (2.2) to a system of ordinary differential equations for the invariant amplitude  $F(\eta)$  defined by

$$\Psi(p; z_1, z_2, z_3) = z_{13}^{-\frac{3}{4h}} (\eta(1 - \eta))^{-\frac{N+1}{Nh}} F(\eta), \quad \eta = \frac{z_{23}}{z_{13}}; \quad (2.12)$$

we find (see Appendix A):

$$\left( h \frac{d}{d\eta} + \frac{\Omega_{12}}{1 - \eta} - \frac{\Omega_{23}}{\eta} \right) F(\eta) = 0, \quad (2.13)$$

$$\Omega_{12} = C_{12} + \frac{1}{N} + 1 = P_{12} + 1, \quad \Omega_{23} = \frac{N - 2}{N} - C_{12} - C_{13} \quad (2.14)$$

( $P_{12}(x \otimes y) = y \otimes x$  and  $F(\eta)$  is an invariant  $SU(N)$  tensor,  $F(\eta) \in \text{Inv}(V_p^* \otimes V^{\otimes 3})$ ). The subspace of invariant tensors in  $V_p^* \otimes V^{\otimes 3}$  is two-dimensional. We shall choose a basis  $I_0, I_1$  in it such that

$$\Omega_{23} I_0 = 0, \quad I_1 = (P_{12} - 1) I_0 \quad (\Rightarrow \Omega_{12} I_1 = 0). \quad (2.15)$$

Setting then

$$F(\eta) = (1 - \eta) f^0(\eta) I_0 + \eta f^1(\eta) I_1, \quad (2.16)$$

we reduce the KZ equation to a system that does not depend on  $N$ :

$$h(1 - \eta) \frac{df^0}{d\eta} = (h - 2)f^0 + f^1, \quad h\eta \frac{df^1}{d\eta} = (2 - h)f^1 - f^0; \quad (2.17)$$

it implies a hypergeometric equation for each  $f^\ell$ :

$$\eta(1 - \eta) \frac{d^2 f^\ell}{d\eta^2} + \left( 1 + \ell - \frac{2}{h} - \left( 3 - \frac{4}{h} \right) \eta \right) \frac{df^\ell}{d\eta} = \left( 1 - \frac{1}{h} \right) \left( 1 - \frac{3}{h} \right) f^\ell, \quad \ell = 0, 1. \quad (2.18)$$

### 3 Dynamical $R$ -matrix exchange relations among CVO

Exchange relations among CVO provide an important ingredient in the finite data characterizing a rational conformal field theory. They determine the spectrum of anomalous dimensions (in other words, they allow a computation of the conformal weights up to

additive integers); they restrict the fusion rules and determine the crossing symmetry (or “duality”) properties of conformal blocks; they allow us to read off the statistics of superselection sectors, which has, according to [21 – 23], an intrinsic meaning in algebraic quantum field theory.

In order to derive the exchange properties of two  $\widehat{su}(N)$  step operators we shall consider the slightly more general matrix element

$$\Psi(p'', p'; z_1, z_2, z_3) = \langle p'' | \varphi(z_1) \otimes \varphi(z_2) \otimes \phi_{p'}(z_3) | 0 \rangle = D_{p''p'}(z_{ab}) F(\eta). \quad (3.1)$$

Here  $p'$  and  $p''$  are the (shifted) weights of  $SU(N)$  IR such that the dimension of the space  $\mathcal{I}_{p''p'} = \text{Inv}(V_{p''}^* \otimes V^{\otimes 2} \otimes V_{p'})$  ( $\ni F(\eta)$ ) is maximal,  $\dim \mathcal{I}_{p''p'} = 2$ , and

$$D_{p''p'}(z_{ab}) = z_{13}^{\Delta(p'') - \Delta(p') - 2\Delta} \eta^{\frac{\Delta(p'') - \Delta(p')}{2} - \frac{\mathbf{p}}{2h} + \frac{2-N^2}{2Nh}} (1 - \eta)^{-\frac{N+1}{Nh}}. \quad (3.2)$$

In (3.2)  $\mathbf{p} = p'_{ij}$  ( $\geq 2$ ) for  $\varphi(z_a)$ ,  $a = 1, 2$ , identified with the CVO  $\varphi_i(z_1)$  and  $\varphi_j(z_2)$ ,  $i < j$ , respectively (for a synopsis of background material concerning  $su(n)$  weights and the corresponding conformal dimensions, see Appendix B). We proceed with a summary of relevant results of [47].

The KZ equation for  $\Psi$  again reduces to the form (2.13), only the expression (2.14) and the relation (A.9) for  $\Omega_{23}$  assume a more general form:

$$\Omega_{23} = \frac{2 - N}{2} + h \frac{\Delta(p'') - \Delta(p')}{2} - C_{12} - C_{13}, \quad \Omega_{23}^2 = \mathbf{p} \Omega_{23}. \quad (3.3)$$

(We recover (2.14) and (A.9) for  $p'_{12} = 2$  ( $= \mathbf{p}$ ),  $p'_{ii+1} = 1$  for  $2 \leq i \leq N - 1$ , in which case  $\Delta(p')$ ,  $\Delta(p'') = \Delta_\phi$  – see (A.2); another simple special case is  $N = 2$ , in which  $p' = p''$ .) The relations (2.15) for the basis  $\{I_0, I_1\}$  of  $SU(N)$  invariants remain unchanged while the hypergeometric system (2.17) assumes the form

$$h(1 - \eta) \frac{df^0}{d\eta} = (h - 2)f^0 + (\mathbf{p} - 1)f^1, \quad h\eta \frac{df^1}{d\eta} = (\mathbf{p} - h)f^1 - f^0. \quad (3.4)$$

A standard basis of (two) solutions is obtained by singling out the possible analytic behaviour of the invariant amplitude  $F(\eta)$  (2.16) for  $\eta \rightarrow 0$ . This gives the so-called *s-channel basis* corresponding, in physical terms, to the *operator product expansion* of  $\varphi_j(z_2)\phi_{p'}(z_3)|0\rangle$  (or of  $\langle 0|\phi_{p''}^*(z_0)\varphi_i(z_1)$  – see Appendix A). In the case at hand, these two solutions,  $(f^\ell(\eta))_\lambda = s_\lambda^\ell(\eta)$ ,  $\lambda = 0, 1$ , are characterized by the property that  $s_0^0(\eta)$  and  $\eta^{-\frac{\mathbf{p}}{h}} s_1^0(\eta)$  are analytic and non-zero at  $\eta = 0$ :

$$s_0^0(0) = K_0, \quad s_1^0(\eta) = K_1 \eta^{\frac{\mathbf{p}}{h}} (1 + O(\eta)), \quad K_\lambda \neq 0, \quad \lambda = 0, 1. \quad (3.5)$$

They are expressed in terms of hypergeometric functions:

$$\begin{aligned} s_0^0(\eta) &= K_0 F\left(1 - \frac{1}{h}, 1 - \frac{\mathbf{p} + 1}{h}; 1 - \frac{\mathbf{p}}{h}; \eta\right), \\ s_1^0(\eta) &= K_0 \frac{1}{\mathbf{p} - h} F\left(1 - \frac{1}{h}, 1 - \frac{\mathbf{p} + 1}{h}; 2 - \frac{\mathbf{p}}{h}; \eta\right), \end{aligned} \quad (3.6)$$



$$\begin{aligned}
s_1^0(\eta) &= K_1 \eta^{\frac{\mathbf{p}}{h}} F\left(1 - \frac{1}{h}, 1 + \frac{\mathbf{p} - 1}{h}; 1 + \frac{\mathbf{p}}{h}; \eta\right), \\
s_1^1(\eta) &= K_1 \frac{\mathbf{p}}{\mathbf{p} - 1} \eta^{\frac{\mathbf{p}}{h} - 1} F\left(-\frac{1}{h}, \frac{\mathbf{p} - 1}{h}; \frac{\mathbf{p}}{h}; \eta\right).
\end{aligned} \tag{3.7}$$

We shall now compute the monodromy representation of the braid group generator  $B(B_1)$  corresponding to the exchange of two “identical particles” 1 and 2. Note first that  $\varphi$  (3.1) is single-valued analytic in the neighbourhood of the real configuration of points  $\{z_1 > z_2 > z_3 > -z_2\}$ . We then choose any path in the homotopy class of

$$\widehat{12}: \quad z_{1,2}(t) = \frac{z_1 + z_2}{2} \pm \frac{1}{2} z_{12} e^{-i\pi t}, \quad z_3(t) = z_3, \quad 0 \leq t \leq 1, \tag{3.8}$$

which thus exchanges  $z_1$  and  $z_2$  in a clockwise direction and perform an analytic continuation of  $\Psi$  along it, followed by a permutation of the  $SU(N)$  indices  $A_1$  and  $A_2$ . This gives

$$z_{12} \rightarrow e^{-i\pi} z_{12}, \quad z_{13} \leftrightarrow z_{13}, \quad 1 - \eta \rightarrow e^{-i\pi} \frac{1 - \eta}{\eta} \quad \left( \eta \rightarrow \frac{1}{\eta} \right), \tag{3.9}$$

$$D_{p''p'}(z_{ab}) \rightarrow \bar{q}^{\frac{N+1}{N}} \eta^{\frac{\mathbf{p}+1}{h}} D_{p''p'}(z_{ab}) \quad \text{for} \quad q^{\frac{1}{N}} = e^{-\frac{i\pi}{Nh}}, \tag{3.10}$$

$$(1 - \eta) I_0 \rightarrow -\frac{1 - \eta}{\eta} (I_0 + I_1), \quad \eta I_1 \rightarrow -\frac{1}{\eta} I_1. \tag{3.11}$$

Using known transformation properties (the “Kummer identities”) for hypergeometric functions (or rederiving them from their integral representations – see [48]), we end up with the braid relation

$$\widehat{12}: \quad D s_\lambda^\ell(\eta) \xrightarrow{\sim} D s_{\lambda'}^\ell(\eta) B_\lambda^{\lambda'}, \quad B = \bar{q}^{\frac{1}{N}} \begin{pmatrix} \frac{q^\mathbf{p}}{[\mathbf{p}]} & K b_\mathbf{p} \\ K^{-1} b_{-\mathbf{p}} & -\frac{\bar{q}^\mathbf{p}}{[\mathbf{p}]} \end{pmatrix}, \tag{3.12}$$

where  $K = \frac{K_1}{K_0}$  and

$$[\mathbf{p}] := \frac{q^\mathbf{p} - \bar{q}^\mathbf{p}}{q - \bar{q}}, \quad b_\mathbf{p} = \frac{\Gamma(1 + \frac{\mathbf{p}}{h}) \Gamma(\frac{\mathbf{p}}{h})}{\Gamma(1 + \frac{\mathbf{p}-1}{h}) \Gamma(\frac{\mathbf{p}+1}{h})} \quad \left( \Rightarrow b_\mathbf{p} b_{-\mathbf{p}} = \frac{[\mathbf{p} + 1][\mathbf{p} - 1]}{[\mathbf{p}]^2} \right). \tag{3.13}$$

The eigenvalues of  $B$  do not depend on the choice of the relative normalization,  $K = K(\mathbf{p})$ , and turn out to be also  $\mathbf{p}$ -independent. Indeed, they are expressed in terms of the dimension  $\Delta = \Delta(\Lambda_1)$  (A.2) of a step operator and by the dimensions  $\Delta_s = \Delta(2\Lambda_1)$  and  $\Delta_a = \Delta(\Lambda_2)$  of the symmetric and skew-symmetric tensor products of two fundamental (“quark”) representations of  $SU(N)$ :

$$\begin{aligned}
\text{spec}(B) &= (e^{i\pi(2\Delta - \Delta_s)}, -e^{i\pi(2\Delta - \Delta_a)}) = (q^{\frac{N-1}{N}}, -\bar{q}^{\frac{N+1}{N}}), \\
\Delta_s &= \frac{(N-1)(N+2)}{Nh}, \quad \Delta_a = \frac{(N+1)(N-2)}{Nh}.
\end{aligned} \tag{3.14}$$

It is remarkable that for  $K(\mathbf{p}) K(-\mathbf{p}) = 1$ , in particular, for

$$K = \frac{\Gamma(\frac{1+\mathbf{p}}{h}) \Gamma(-\frac{\mathbf{p}}{h})}{\Gamma(\frac{1-\mathbf{p}}{h}) \Gamma(\frac{\mathbf{p}}{h})} \rho(\mathbf{p}), \quad \rho(\mathbf{p}) \rho(-\mathbf{p}) = 1 \quad (= K(\mathbf{p}) K(-\mathbf{p})), \tag{3.15}$$

(3.12) agrees with the (dynamical)  $R$ -matrix exchange relations

$$\varphi_i^B(z_2) \overset{\curvearrowright}{\varphi_j^A(z_1)} = \varphi_s^A(z_1) \varphi_t^B(z_2) \hat{R}(\mathbf{p})_{ij}^{st} \quad (3.16)$$

linked in [38] with the properties of an intertwining quantum matrix algebra generated by an  $N \times N$  matrix  $(a_\alpha^i)$  with non-commuting entries and by  $N$  commuting unitary operators  $q^{p_i}$  ( $\prod_{i=1}^N q^{p_i} = 1$ ), such that

$$q^{p_i} a_\alpha^j = a_\alpha^j q^{p_i + \delta_i^j - \frac{1}{N}}, \quad \hat{R}(p) a_1 a_2 = a_1 a_2 \hat{R}, \quad \varphi_\alpha^A(z) = \varphi_i^A(z) a_\alpha^i. \quad (3.17)$$

Here  $\hat{R} = (\hat{R}_{\beta_1 \beta_2}^{\alpha_1 \alpha_2})$  and  $\hat{R}(p) = (\hat{R}(p)_{j_1 j_2}^{i_1 i_2})$  are the  $U_q(sl_N)$  and the dynamical  $R$ -matrices, respectively, multiplied by a permutation,  $\hat{R} = R P$  and we are using Faddeev's concise notation for tensor products.  $\hat{R}(p)$  obeys the Gervais–Neveu [49] “dynamical Yang–Baxter equation” whose general solution satisfying “the *ice condition*” (the condition that  $\hat{R}(p)_{kl}^{ij}$  vanishes unless the *unordered* pairs  $(i, j)$  and  $(k, l)$  coincide) was found by A. Isaev [50]. Its  $2 \times 2$  block

$$\begin{pmatrix} \hat{R}(p)_{ij}^{ij} & \hat{R}(p)_{ji}^{ij} \\ \hat{R}(p)_{ij}^{ji} & \hat{R}(p)_{ji}^{ji} \end{pmatrix} = \bar{q}^{\frac{1}{N}} \begin{pmatrix} \frac{q^{\mathbf{p}}}{[\mathbf{p}]} & \frac{[\mathbf{p}-1]}{[\mathbf{p}]} \rho(\mathbf{p}) \\ \frac{[\mathbf{p}+1]}{[\mathbf{p}]} \rho(-\mathbf{p}) & -\frac{1}{[\mathbf{p}]} \end{pmatrix} \quad (\rho(\mathbf{p})\rho(-\mathbf{p}) = 1) \quad (3.18)$$

indeed coincides with  $B$ , for  $K$  given by (3.15).

The monodromy representations of the braid group in the space of solutions of the KZ equation was first studied systematically in [25]. The Drinfeld–Kohno theorem [10, 11] (see also [41], Ch. 19) says, essentially, that for generic  $q$  this monodromy representation is always given by (a finite-dimensional representation of) Drinfeld's universal  $R$ -matrix. In the physically interesting case of  $q$ , an (even) root of unity ( $q^h = -1$ ), the situation is more complicated. A problem already appears in Eq. (3.18): for  $\mathbf{p} = h$ ,  $[h] = 0$  and the right-hand side of (3.18) makes no sense. In fact, the representation of the braid group is not unitarizable for such values of  $\mathbf{p}$ . The corresponding “unphysical” solutions of the KZ equation cannot, on the other hand, be thrown away by decree; otherwise, the chiral field algebra will not be closed under multiplication.

It turns out that a monodromy representation of the braid group can, in fact, be defined on the entire space of solutions of the KZ equation. It is, in general, indecomposable. The above  $s$ -channel basis, however, does not extend to  $\mathbf{p} = h$  (as is manifest in Eq. (3.6)).

## 4 Regular basis of solutions of the KZ equation and Schwarz finite monodromy problem

It follows from (3.16) and (3.17) that the chiral fields (unlike the CVO  $\varphi_j^A$ ) satisfy  $\mathbf{p}$ -independent (and hence, non-singular) exchange relations:

$$\begin{aligned} \varphi_\alpha^B(z_2) \overset{\curvearrowright}{\varphi_\beta^A(z_1)} &= \varphi_\rho^A(z_1) \varphi_\sigma^B(z_2) \hat{R}_{\alpha\beta}^{\rho\sigma}, \\ \hat{R} &= \bar{q}^{\frac{1}{N}} (q \mathbb{I} - A), \quad A_{\alpha\beta}^{\rho\sigma} = q^{\epsilon_{\sigma\rho}} \delta_\sigma^\rho \delta_\beta^\sigma - \delta_\beta^\rho \delta_\alpha^\sigma, \end{aligned} \quad (4.1)$$

$$\epsilon_{\sigma\rho} = \begin{cases} 1, & \sigma > \rho \\ 0, & \sigma = \rho \\ -1, & \sigma < \rho \end{cases}, \quad A^2 = [2] A, \quad [2] = q + \bar{q}. \quad (4.2)$$

The singularity in the conformal block (3.6) for  $\mathfrak{p} = h$  is thus a consequence of the introduction of CVO which pretend to diagonalize the (in general, non-diagonalizable) monodromy matrix  $M$  defined by  $\varphi(z e^{2\pi i}) = \varphi M$ . A regular basis of conformal blocks is linked to a regular basis in  $U_q(sl_N)$  invariant tensors (with respect to the indices  $\alpha, \beta, \dots$ ). Such a basis has been introduced for  $N = 2$  in [51] and recently generalized to four-point blocks involving a pair of  $U_q(sl_N)$  step operators [47]. Its counterpart in the space of conformal blocks of the  $SU(N)$  WZNW model was written down in [48] (for  $N = 2$ ) and in [47] for arbitrary  $N$ . We shall display here a regular basis of four-point conformal blocks  $f_\lambda^\ell$ , only mentioning in conclusion some properties of their quantum group counterparts  $\mathcal{I}^\lambda$ .

Writing the Möbius-invariant amplitude (2.16) in the form

$$F(\eta) = F_0(\eta) \mathcal{I}^0 + F_1(\eta) \mathcal{I}^1, \quad F_\lambda(\eta) = (1 - \eta) f_\lambda^0(\eta) I_0 + \eta f_\lambda^1(\eta) I_1, \quad (4.3)$$

where  $\mathcal{I}^\lambda$  are  $U_q(sl_N)$  invariant tensors to be specified below, we define the regular basis by

$$B\left(\frac{\mathfrak{p}-1}{h}, \frac{2}{h}\right) f_0^\ell(\eta) = \int_\eta^1 t^{\frac{\mathfrak{p}-1}{h}-\ell} (1-t)^{\frac{1}{h}-1+\ell} (t-\eta)^{\frac{1}{h}-1} dt = \quad (4.4)$$

$$= B\left(\frac{1}{h}, \ell + \frac{1}{h}\right) (1-\eta)^{\frac{2}{h}-1+\ell} F\left(\ell - \frac{\mathfrak{p}-1}{h}, \ell + \frac{1}{h}; \ell + \frac{2}{h}; 1-\eta\right),$$

$$B\left(\frac{\mathfrak{p}-1}{h}, \frac{2}{h}\right) f_1^\ell(\eta) = \int_0^\eta t^{\frac{\mathfrak{p}-1}{h}-\ell} (1-t)^{\frac{1}{h}-1+\ell} (\eta-t)^{\frac{1}{h}-1} dt = \quad (4.5)$$

$$= B\left(\frac{1}{h}, 1 - \ell + \frac{\mathfrak{p}-1}{h}\right) \eta^{\frac{\mathfrak{p}}{h}-\ell} F\left(1 - \ell - \frac{1}{h}, 1 - \ell + \frac{\mathfrak{p}-1}{h}; 1 - \ell + \frac{\mathfrak{p}}{h}; \eta\right)$$

$\left(B(\mu, \nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\right)$ . A direct computation using the integral representations for  $f_\lambda^\ell(\eta)$  yields the following form for the braid matrix  $B_1$ , exchanging the arguments 1 and 2 (the counterpart of  $B$  (3.12) in the regular basis,  $B$  and  $B_1$  having the same eigenvalues):

$$B_1 = \bar{q}^{\frac{1}{N}} \begin{pmatrix} q & 1 \\ 0 & -\bar{q} \end{pmatrix}, \quad \det(q^{\frac{1}{N}} B_1) = -1 = \det(q^{\frac{1}{N}} B), \quad \text{tr}(q^{\frac{1}{N}} B_1) = q - \bar{q} = \text{tr}(q^{\frac{1}{N}} B). \quad (4.6)$$

$B_1$  and  $B$  are thus related by a similarity transformation whenever both make sense:

$$B_1 = S B S^{-1}, \quad S = \begin{pmatrix} 1 & 0 \\ -\frac{[\mathfrak{p}-1]}{[\mathfrak{p}]} & \rho(\mathfrak{p}) \frac{[\mathfrak{p}-1]}{[\mathfrak{p}]} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 0 \\ \rho(-\mathfrak{p}) & \rho(-\mathfrak{p}) \frac{[\mathfrak{p}]}{[\mathfrak{p}-1]} \end{pmatrix}. \quad (4.7)$$

Similarly, the exchange matrix  $B_2$  (corresponding to the braiding  $\tilde{2} \tilde{3}$ ) is given by

$$B_2 = S q^{\frac{1-\mathfrak{p}}{N}} \begin{pmatrix} -\bar{q} & 0 \\ 0 & q^{\mathfrak{p}-1} \end{pmatrix} S^{-1} = q^{\frac{1-\mathfrak{p}}{N}} \begin{pmatrix} -\bar{q} & 0 \\ \frac{q^{2(\mathfrak{p}-1)}-1}{q^{\mathfrak{p}-1}} & q^{\mathfrak{p}-1} \end{pmatrix}. \quad (4.8)$$

The eigenvalues of  $B_2$ , like those of  $B_1$ , cf. (3.14), are expressed in terms of conformal dimensions. Setting  $i = 1$ ,  $j = 2$ , assuming that  $p'$  is the symmetric tensor representation (B.3) and

$$p_s = (\mathbf{p} + 1, 1, \dots, 1), \quad p_a = (\mathbf{p} - 1, 2, \dots, 1) \quad (4.9)$$

(in the notation  $p = (p_{12}, p_{23}, \dots, p_{N-1N})$ ), we find

$$e^{i\pi(\Delta + \Delta(p) - \Delta(p_a))} = q^{\frac{1-\mathbf{p}-N}{N}}, \quad e^{i\pi(\Delta + \Delta(p) - \Delta(p_s))} = q^{(\mathbf{p}-1)\frac{N-1}{N}} \quad (4.10)$$

for

$$\Delta(p_a) = \frac{(N-1)(\mathbf{p}-2)^2 + (N-2)(N+1)\mathbf{p}}{2Nh}, \quad \Delta(p_s) = \frac{(N-1)\mathbf{p}(\mathbf{p}+N)}{2Nh}. \quad (4.11)$$

We observe that, unlike  $B$ , the matrices  $B_1$  and  $B_2$  are defined for  $0 < \mathbf{p} < 2h$ . The singularity in  $S$  (4.7), as well as the non-existence of the  $s$ -channel basis for  $\mathbf{p} = h$ , is due to the simple fact that the matrix  $B_2$  (4.8) is non-diagonalizable in this case (while the  $s$ -channel basis could be defined as “the basis in which  $B_2$  is diagonal”). Note that, for  $\mathbf{p} = 2$ ,  $B_2$  becomes similar to  $B_1$ :

$$B_2 = \bar{q}^{\frac{1}{N}} \begin{pmatrix} -\bar{q} & 0 \\ 1 & q \end{pmatrix} = \sigma_1 B_1 \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{for } \mathbf{p} = 2) \quad (4.12)$$

and  $B_1$  and  $B_2$  generate a representation of the braid group  $\mathcal{B}_3$  with central element  $c^3 = (B_1 B_2)^3 = \bar{q}^{\frac{6}{N}} \mathbb{I}$ .

Whenever the  $s$ -channel basis (3.6), (3.7) exists, it is related to the regular basis (4.3)–(4.5) by

$$F_0(\eta) \mathcal{I}^0 + F_1(\eta) \mathcal{I}^1 = s_0(\eta) \mathcal{S}^0 + s_1(\eta) \mathcal{S}^1. \quad (4.13)$$

Here  $\mathcal{I}^0$  and  $\mathcal{S}^0$  are equal and can be expressed as a matrix element of a product of  $a_\alpha^i$  satisfying (3.18):

$$(\mathcal{S}^0 = \mathcal{I}^0) = \langle p'' | a_{\alpha_1}^i a_{\alpha_2}^j | p' \rangle \quad \text{for } i < j; \quad \mathcal{S}^1 = \langle p'' | a_{\alpha_1}^j a_{\alpha_2}^i | p' \rangle. \quad (4.14)$$

(If  $p'$  is the symmetric tensor representation, see (B.3), then we choose  $i = 1$ ,  $j = 2$ .) The invariant tensor  $\mathcal{I}^1$ , on the other hand, is related to  $\mathcal{I}^0$  by

$$\mathcal{I}_{\dots\alpha_1\alpha_2\dots}^1 = -\mathcal{I}_{\dots\sigma_1\sigma_2\dots}^0 A_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}, \quad (4.15)$$

where  $A$  is the quantum antisymmetrizer defined in (4.2). The exchange relations (3.17) with the dynamical  $R$ -matrix (3.18) then allow us to relate  $\mathcal{S}^\lambda$  with  $\mathcal{I}^\lambda$  and conversely:

$$\rho(\mathbf{p}) \mathcal{S}_{\alpha\beta}^1 = \mathcal{I}_{\alpha\beta}^0 + \frac{[\mathbf{p}]}{[\mathbf{p}-1]} \mathcal{I}_{\alpha\beta}^1, \quad \mathcal{I}_{\alpha\beta}^1 = \frac{[\mathbf{p}-1]}{[\mathbf{p}]} (\rho(\mathbf{p}) \mathcal{S}_{\alpha\beta}^1 - \mathcal{S}_{\alpha\beta}^0). \quad (4.16)$$

For  $\mathbf{p} (= p'_{ij}) = h$ ,  $\mathcal{S}^0$  and  $\mathcal{S}^1$  are proportional,  $\mathcal{S}^0 = \rho(h) \mathcal{S}^1$ , so that they do not form a basis;  $\mathcal{I}^1$ , on the other hand, is defined unambiguously by (4.15) and is linearly independent of  $\mathcal{I}^0$ .

The above regular basis also has a remarkable number theoretic property: the matrix elements of  $q^{\frac{1}{N}}B_1$  (and of  $q^{\frac{p-1}{N}}B_2$ ) belong to the *cyclotomic field*  $\mathbb{Q}(q)$  of polynomials in  $q$  with rational coefficients for  $q^h = -1$ . This fact has been used in [52] to classify all cases in which the monodromy representation of the braid group  $\mathcal{B}_4$ , for  $N = 2$ , is a finite matrix group or, equivalently, the cases in which the KZ equation has an algebraic solution (a classical problem solved for the hypergeometric equation by H.A. Schwarz in the 1870's). As pointed out in [47], this result readily extends to higher  $N$  in the case of three step operators (for  $\mathcal{B}_3$ ). The argument uses one of the oldest and most beautiful concepts in group theory, the Galois group, so that it deserves to be summarized.

The space of  $U_q(sl_N)$  invariants admits a braid invariant hermitean form  $(\cdot, \cdot)$ . In the regular basis,  $Q^{\lambda\mu} \equiv (\mathcal{I}^\lambda, \mathcal{I}^\mu)$  belong to the real subfield  $\mathbb{Q}([2]) = \mathbb{Q}(q + \bar{q})$  of  $\mathbb{Q}(q)$ . The special case of  $N = 2$  is worked out in Appendix C. In that case the resulting hermitean form  $Q$  is positive-semidefinite for  $q = e^{\pm i\frac{\pi}{h}}$  and has a kernel of dimension  $2\mathfrak{p} - h$  for  $2\mathfrak{p} > h$ . For the case  $\mathfrak{p} = 2$  of interest this kernel is only nontrivial at level 1, for  $h = 3$ , when it is one-dimensional.

We define a *primitive root* of the equation  $q^h = -1$  as any zero of the irreducible element  $P_h(q)$  of the ring of polynomials with integer coefficients satisfying  $P_h(e^{\pm i\frac{\pi}{h}}) = 0$ . (There is a unique such irreducible polynomial with coefficient to the highest power of  $q$ , equal to 1.) The *Galois group*  $\text{Gal}_h$  for  $P_h$ , the group that permutes its roots, consists of all substitutions of the form

$$\text{Gal}_h = \{ q \rightarrow q^\ell, \ 0 < \ell < 2h, \ (\ell, 2h) = 1 \} \quad (4.17)$$

(in the last condition in the definition we use the familiar notation  $(\ell, m)$  for the greatest common divisor of  $\ell$  and  $m$ ).

A hermitean form with entries in a cyclotomic field  $\mathbb{Q}(q)$  is called *totally positive* if all its Galois transforms are positive. Our analysis is based on the following theorem. The total positivity of a  $\mathcal{B}_n$ -invariant form  $Q$  is sufficient (and, if the invariant form is unique, also necessary) for the monodromy representation of  $\mathcal{B}_n$  in  $q\text{-Inv}(V^{\otimes n})/\text{Ker } Q$  to be a finite matrix group. For  $N = 2$  and  $h > 3$  we find [52] that the total positivity of  $Q$  is equivalent to the total positivity of the quantum dimension  $[3] = \frac{q^3 - \bar{q}^3}{q - \bar{q}} \equiv q^2 + 1 + \bar{q}^2$  encountered in the tensor product expansion of the tensor square of the two-dimensional representation:  $[2]^2 = 1 + [3]$ . This amounts to finding the values of  $h \geq 4$  such that

$$1 + \cos \frac{2\pi\ell}{h} > 0 \quad \text{for} \quad (\ell, 2h) = 1. \quad (4.18)$$

The only solutions are  $h = 4, 6, 10$ . If we add to these the case  $h = 3$ , in which the commutator subgroup of  $\mathcal{B}_4$  is trivial ( $B_0 B_1 B_0^{-1} B_1^{-1} = 1 = B_1 B_2 B_1^{-1} B_2^{-1} = \dots$ ), we see that the four cases of “finite monodromy” correspond to the four integral quadratic algebras of dimension  $h - 2 = 1, 2, 4, 8$ .

## Acknowledgements

I.T. thanks John Roberts for an enlightening correspondence concerning the scope of the main theorem of [19] and George Pogosyan for his invitation to the Group Theoretical

Methods Colloquium. The authors thank the Theory Division of CERN for hospitality and acknowledge partial support from the Bulgarian National Council for Scientific Research under contract F-828.

## Appendix A. Reduction of the KZ equation for $SU(N)$ step operators to an $N$ -independent system of hypergeometric equations

The “wave function”  $\Psi(p; z_1, z_2, z_3)$  can be viewed as a  $z_0 \rightarrow \infty$  limit of a Möbius and  $SU(N)$ -invariant four-point function

$$w(z_0; z_1, z_2, z_3) = \langle 0 | \phi^*(z_0) \otimes \varphi(z_1) \otimes \varphi(z_2) \otimes \varphi(z_3) | 0 \rangle, \quad \phi(0) | 0 \rangle \sim \boxplus. \quad (\text{A.1})$$

The step operator  $\varphi$  and the field  $\phi$  have  $su(N)$  weights  $\Lambda_1$  and  $\Lambda_1 + \Lambda_2$  ( $\Lambda_j$ ,  $j = 1, \dots, N-1$  being the fundamental  $su(N)$  weights). Their conformal dimensions are

$$\begin{aligned} \Delta &= \Delta(\Lambda_1) = \frac{1}{2h} C_2(\Lambda_1) = \frac{N^2 - 1}{2hN}, \\ \Delta_\phi &= \Delta(\Lambda_1 + \Lambda_2) = \frac{1}{2h} C_2(\Lambda_1 + \Lambda_2) = \frac{3}{2hN} (N^2 - 3), \end{aligned} \quad (\text{A.2})$$

where  $C_2(\Lambda)$  stands for the eigenvalue of the second-order Casimir invariant (normalized in such a way that for the adjoint representation  $C_2(\Lambda_1 + \Lambda_{N-1}) = 2N$ ). Möbius (i.e.  $SL(2)$ ) invariance implies that we can write  $w$  in the form

$$w(z_0; z_1, z_2, z_3) = D_N(z_{ab}) F(\eta), \quad \eta = \frac{z_{01} z_{23}}{z_{02} z_{13}}. \quad (\text{A.3})$$

The prefactor  $D_N$  is a product of powers of the coordinate differences  $z_{ab}$  determined from infinitesimal Möbius invariance

$$\left( z_0^\nu \left( z_0 \frac{\partial}{\partial z_0} + (\nu + 1) \Delta_\phi \right) + \sum_{c=1}^3 z_c^\nu \left( z_c \frac{\partial}{\partial z_c} + (\nu + 1) \Delta \right) \right) D_N(z_{ab}) = 0 \quad (\text{A.4})$$

for  $\nu = 0, \pm 1$ , up to powers of  $\eta$  and  $(1 - \eta)$ , which are fixed by requiring that there exists a solution  $F(\eta)$  of the resulting ordinary differential equation that takes finite non-zero values for both  $\eta = 0$  and  $\eta = 1$ :

$$D_N(z_{ab}) = \left( \frac{z_{13}^{2N+4}}{z_{03}^{3N+5} z_{12}} \right)^{\frac{N-2}{2Nh}} \frac{(1 - \eta)^{\frac{(N-4)(N+1)}{2Nh}}}{z_{01}^{\frac{N^2+N-3}{Nh}} z_{23}^{\frac{N+1}{Nh}}} = \left( \frac{z_{13}^{-3} (\eta(1 - \eta))^{-N-1}}{z_{02}^{N^2-1} (z_{01} z_{03})^{N^4-4}} \right)^{\frac{1}{Nh}}. \quad (\text{A.5})$$

Comparing the last expression with (2.12) we find the relation

$$\begin{aligned} \Psi_p(z_1, z_2, z_3) &= \lim_{z_0 \rightarrow \infty} \{ z_0^{2\Delta_\phi} w(z_0; z_1, z_2, z_3) \} = \\ &= \left( z_{02}^{N^2-1} (z_{01} z_{03})^{N^4-4} \right)^{\frac{1}{Nh}} w(z_0; z_1, z_2, z_3). \end{aligned} \quad (\text{A.6})$$

Applying to the 4-point function  $w$  (A.3) the covariant derivative  $h\nabla_1$  (2.4),

$$h\nabla_1 = h \frac{\partial}{\partial z_1} + \frac{C_{01}}{z_{01}} - \frac{C_{12}}{z_{12}} - \frac{C_{13}}{z_{13}}, \quad (\text{A.7})$$

and using the  $SU(N)$  invariance condition

$$(C_{01} + C_{12} + C_{13} + C_2(\Lambda_1)) w(z_0; z_1, z_2, z_3) = 0, \quad (\text{A.8})$$

we end up with (2.13) for  $\Omega_{ab}$  given by (2.14). The operators  $\Omega_{12}$  and  $\Omega_{23}$  have an algebraic characterization of the Temperley–Lieb type (see [47], Eq. (2.14)):

$$\Omega_{12}\Omega_{23}\Omega_{12} = \Omega_{12}, \quad \Omega_{23}\Omega_{12}\Omega_{23} = \Omega_{23}, \quad \Omega_{ab}^2 = 2\Omega_{ab}. \quad (\text{A.9})$$

In particular, each  $\Omega_{ab}$  has eigenvalues 0 and 2. If we regard  $\phi^*$  as a mixed tensor of  $2N - 3$  indices,  $\phi^* = \{(\phi^*)^{B_1 \dots B_{N-1} C_1 \dots C_{N-2}}\}$ , then the  $SU(N)$  invariant tensors  $I_0$  and  $I_1$  of Eq. (2.15) can be presented in the form

$$I_0 = (\epsilon^{B_1 \dots B_{N-1} A_1} \epsilon^{C_1 \dots C_{N-2} A_2 A_3}), \quad I_1 = (P_{12} - 1) I_0, \quad (\text{A.10})$$

where  $\epsilon$  is the totally antisymmetric Levi-Civita tensor, and  $P_{12}$  permutes the indices  $A_1$  and  $A_2$ . In this basis the operators  $\Omega_{12}$  and  $\Omega_{23}$  have the following matrix realization:

$$\Omega_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Omega_{23} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \quad (\text{A.11})$$

i.e.  $\Omega_{12}I_0 = 2I_0 + I_1$ ,  $\Omega_{23}I_1 = I_0 + 2I_1$ , etc. Remarkably, the relations (A.9) and (A.11) are independent of  $N$ . Inserting (2.16) into (2.13) and using (A.11), we thus end up with the  $N$ -independent system (2.17) of a hypergeometric type.

## Appendix B. Shifted $SU(N)$ weights and CVO. Symmetric tensor representations

If  $\Lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i$ ,  $\lambda_i \in \mathbb{Z}_+$ , is an  $su(N)$  *highest-weight* ( $\lambda_i$  being the number of columns of height  $i$  in the associated Young tableau), then the corresponding *shifted weight* is written in terms of barycentric coordinates  $p = (p_1, \dots, p_N)$  as follows:

$$p = \Lambda + \rho = \sum_{i=1}^{N-1} p_{i+1} \Lambda_i, \quad p_{ij} = p_i - p_j, \quad p_{i+1} = \lambda_i + 1, \quad \sum_{i=1}^N p_i = 0 \quad (\text{B.1})$$

( $\rho = \sum_{i=1}^{N-1} \Lambda_i$  is the half sum of the positive roots.) The conformal dimension of a  $\widehat{su}(N)$  primary field of weight  $p$  is expressed in terms of the second-order Casimir operator  $C_2(p)$  :

$$2h\Delta(p) = C_2(p) = \frac{1}{N} \sum_{i < j} (p_{ij}^2 - (j - i)^2) = \frac{1}{N} \sum_{i < j} p_{ij}^2 - \frac{N(N^2 - 1)}{12}. \quad (\text{B.2})$$

The CVO  $\varphi_j(= \varphi_j^A(z))$  is related to the  $U_q(sl_N)$  covariant field  $\varphi_\alpha(= \varphi_\alpha^A(z))$  by (3.17).

In the example of a symmetric tensor representation  $p'$  and its counterpart  $p''$  defined by the requirement  $\dim \mathcal{I}_{p'p''} = 2$  we have

$$\begin{aligned} p'_{12} &= p, \quad p'_{ii+1} = 1, \quad 2 \leq i \leq N-1, \\ C_2(p') &= \frac{N-1}{N}(p-1)(p+N-1), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} p''_{12} &= p, \quad p''_{23} = 2, \quad p''_{ii+1} = 1, \quad 3 \leq i \leq N-1, \\ C_2(p'') &= (p+1) \frac{N^2 + (p-2)N - (p+1)}{N}. \end{aligned} \quad (\text{B.4})$$

The dimensions of these representations are expressed in terms of binomial coefficients:

$$d(p') = \binom{p+N-2}{N-1}, \quad d(p'') = p \binom{p+N-1}{p+1}. \quad (\text{B.5})$$

In computing the prefactor (3.2) one needs

$$\begin{aligned} \Delta(p'') - \Delta(p') - 2\Delta &= \\ &= \frac{(N-2)(N+p)}{Nh} - \frac{N^2-1}{Nh} = \frac{(p-2)(N-2)-3}{Nh}, \end{aligned} \quad (\text{B.6})$$

$$\frac{\Delta(p'') - \Delta(p')}{2} - \frac{p}{2h} - \frac{N^2-2}{2Nh} = -\frac{N+p-1}{Nh}. \quad (\text{B.7})$$

## Appendix C. Basis of $U_q(sl_2)$ invariants in $V^{\otimes 4}$ for $V = \mathbb{C}^2$ . Braid-invariant hermitean form

The basic  $U_q(sl_N)$  invariant in  $V^{\otimes N}$  for  $V = \mathbb{C}^N$  is the  $q$ -deformed Levi-Civita tensor

$$\mathcal{E}_{\alpha_1 \dots \alpha_N} = \bar{q}^{\frac{1}{2} \binom{N}{2}} (-q^2)^\ell \begin{pmatrix} N & \dots & 1 \\ \alpha_1 & \dots & \alpha_N \end{pmatrix}, \quad (\text{C.1})$$

where  $\ell$  is the length of the permutation  $\begin{pmatrix} N & \dots & 1 \\ \alpha_1 & \dots & \alpha_N \end{pmatrix}$ , i.e. the minimal number of transpositions of neighbouring indices; in particular, for  $N = 2$

$$(\mathcal{E}_{\alpha_1 \alpha_2}) = \begin{pmatrix} 0 & -q^{\frac{1}{2}} \\ \bar{q}^{\frac{1}{2}} & 0 \end{pmatrix}, \quad \text{i.e. } \mathcal{E}_{21} = \bar{q}^{\frac{1}{2}}, \quad \mathcal{E}_{12} = -q^{\frac{1}{2}}. \quad (\text{C.2})$$

The regular basis of  $U_q(sl_2)$  invariants in  $V^{\otimes 4} = (\mathbb{C}^2)^{\otimes 4}$  is

$$\mathcal{I}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^0 = \mathcal{E}_{\alpha_1 \alpha_2} \mathcal{E}_{\alpha_3 \alpha_4}, \quad \mathcal{I}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^1 = \mathcal{E}_{\alpha_1 \alpha_4} \mathcal{E}_{\alpha_2 \alpha_3}. \quad (\text{C.3})$$

Their inner products are given by traces:

$$\begin{aligned} (\mathcal{I}^\lambda, \mathcal{I}^\mu) &= \sum_{\alpha_1 \dots \alpha_4} \mathcal{I}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^\lambda \mathcal{I}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^\mu, \\ (\mathcal{I}^\lambda, \mathcal{I}^\lambda) &= [2]^2, \quad \lambda = 0, 1, \quad (\mathcal{I}^0, \mathcal{I}^1) = -[2]. \end{aligned} \quad (\text{C.4})$$



To verify braid invariance, note that

$$\begin{aligned} B_{1\lambda}^0 \mathcal{I}^\lambda &= q^{\frac{1}{2}} \mathcal{I}^0 + \bar{q}^{\frac{1}{2}} \mathcal{I}^1, & B_{1\lambda}^1 \mathcal{I}^\lambda &= -\bar{q}^{\frac{3}{2}} \mathcal{I}^1, \\ (B_{1\lambda}^0 \mathcal{I}^\lambda, B_{1\mu}^0 \mathcal{I}^\mu) &= 2[2]^2 - (q + \bar{q})[2] = [2]^2 = (\mathcal{I}^0, \mathcal{I}^0), \text{ etc.} \end{aligned} \quad (\text{C.5})$$

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